The (p,k) numerical range and quantum error correction

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The Hong Kong Polytechnic University

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Guelph, Canada

This is based on a joint work with
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Mikio Nakahara (Kinki University)
Yiu-Tung Poon (Iowa State University)

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Outline

- Rank-k numerical range vs QECC
- (p, k) numerical range vs OQEC

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Quantum error correction

A quantum channel $\mathcal{E} : M_n \to M_n$ is a completely positive, trace preserving linear map of the form

$$\mathcal{E} : \rho \mapsto \sum_{j=1}^{r} F_j \rho F_j^\dagger \quad \text{with} \quad \sum_j F_j^\dagger F_j = I.$$ [Choi, LAA 10:285-290 (1975)]

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Three qubit bit-flip channel **[Nakahara, Ohmi, CRC press, 2008]**

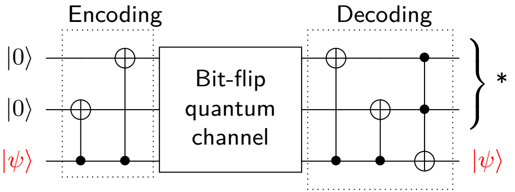
Each qubit flips independent with a probability $p << 1$. Further, assume that at most one of qubits can be flipped. Mathematically, the three-qubit bit-flip channel $\mathcal{E} : M_8 \to M_8$ is defined by

$$\mathcal{E}(\rho) = \sum_{j=1}^{4} F_j \rho F_j^\dagger,$$

with error operators

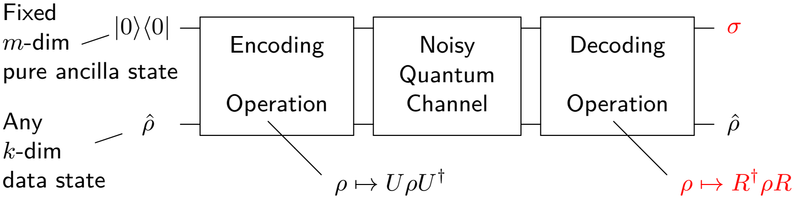
$$F_1 = \sqrt{p_1}\, I \otimes I \otimes I,$$
$$F_2 = \sqrt{p_2}\, X \otimes I \otimes I,$$
$$F_3 = \sqrt{p_3}\, I \otimes X \otimes I,$$
$$F_4 = \sqrt{p_4}\, I \otimes I \otimes X.$$

where $\sum_{j=1}^{4} p_j = 1$.



[Nakahara, Tomita arXiv:1101.0413 (2011)]

Quantum Error Correction Code (QECC)



$$\mathcal{E}\left(U\left(|0\rangle\langle 0| \otimes \hat{\rho}\right)U^\dagger\right) = R\left(\sigma \otimes \hat{\rho}\right)R^\dagger \qquad \forall \hat{\rho} \in M_k$$

$$\updownarrow$$

$$R^\dagger(\mathcal{E}(\rho))R = \rho \qquad \forall P_{\mathbf{V}}\rho P_{\mathbf{V}} = \rho$$

In the original Knill-Laflamme result, a recovery channel is needed.

An orthogonal projection of $\mathbb{C}^n$ to a k dimensional subspace $\mathbf{V}$

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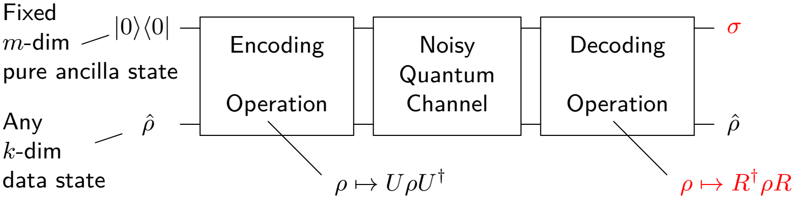

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Quantum Error Correction Code [Knill and Laflamme, PRA 55:900-911 (1997)]

A subspace $\mathbf{V}$ of $\mathbb{C}^n$ is a QECC for $\mathcal{E}$ if and only if

$$P_{\mathbf{V}} F_i^\dagger F_j P_{\mathbf{V}} = \lambda_{ij} P_{\mathbf{V}} \quad \text{for all} \quad 1 \le i, j \le r.$$

[Li, Nakahara, Poon, S., Tomita, QIC 12:149-158 (2012)]

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Rank-k numerical range

In connection to Quantum Error Correction, Choi, et al suggested

[Choi, Kribs, and Zyczkowski LAA 418:828-839 (2006)]

Joint rank-k numerical range

The joint rank-k numerical range of $\mathcal{A} = (A_1, \ldots, A_m)$ with A_j on $\mathcal{B}(\mathcal{H})$ is defined by

$$\Lambda_k(\mathcal{A}) = \{(\mu_1, \ldots, \mu_m) \in \mathbb{C}^m : PA_jP = \mu_j P \text{ for some rank-}k \text{ orthogonal projection } P\}.$$

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- $\mathcal{E}$ is correctable if and only if $\Lambda_k(E_1^\dagger E_1, E_1^\dagger E_2, \ldots, E_m^\dagger E_m) \neq \emptyset$.

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- $\mathcal{E}$ is correctable if and only if $\Lambda_k(E_1^\dagger E_1, E_1^\dagger E_2, \ldots, E_m^\dagger E_m) \neq \emptyset$.
- In particular, for a bi-unitary channel

$$\mathcal{E} : \rho \mapsto tU_1\rho U_1^\dagger + (1-t)U_2\rho U_2^\dagger$$

$\mathcal{E}$ is correctable if and only if $\Lambda_k(U_1^\dagger U_2) \neq \emptyset$.

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Rank-k numerical range

Rank-k numerical range [Choi, Kribs, and Zyczkowski LAA 418:828-839 (2006)]

The rank-k numerical range of A on $\mathcal{B}(\mathcal{H})$ is defined by

$$\Lambda_k(A) = \{\mu \in \mathbb{C} : PAP = \mu P \text{ for some rank-}k\text{ orthogonal projection P}\}.$$

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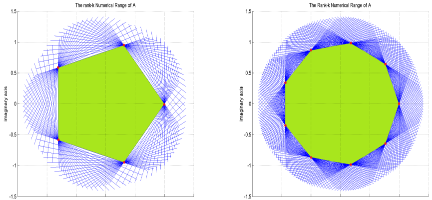
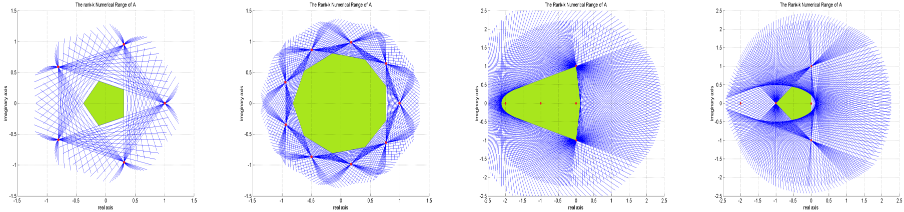


[Choi, Giesinger, Holbrook, Kribs, LAMA 56:53-64 (2008)]
[Choi, Holbrook, Kribs, Zyczkowski, OAM 1:409-426 (2007)]
[Choi, Kribs, Zyczkowski, RMP 58:77-91 (2006)]
[Li, Poon, S., JMAA 348:843-855 (2008)]
[Li, Poon, S., LAMA 57:365-368 (2009)]
[Li, S., PAMS 136:3013-3023 (2008)]
[Woerdeman, LAMA 56:65-67 (2008)]



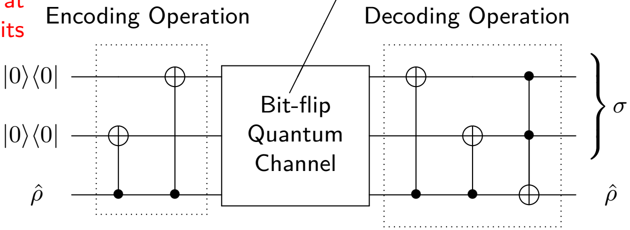
Bit-flip Quantum Channel

Three qubit bit-flip quantum channel

$$\mathcal{E} : \rho \mapsto F_1 \rho F_1^\dagger + F_2 \rho F_2^\dagger + F_3 \rho F_3^\dagger + F_4 \rho F_4^\dagger$$

$$\begin{aligned} F_1 &= \sqrt{q_1}\, I \otimes I \otimes I \\ F_2 &= \sqrt{q_2}\, X \otimes I \otimes I \\ F_3 &= \sqrt{q_3}\, I \otimes X \otimes I \\ F_4 &= \sqrt{q_4}\, I \otimes I \otimes X \end{aligned}$$

Assume that at most one of qubits can be flipped!



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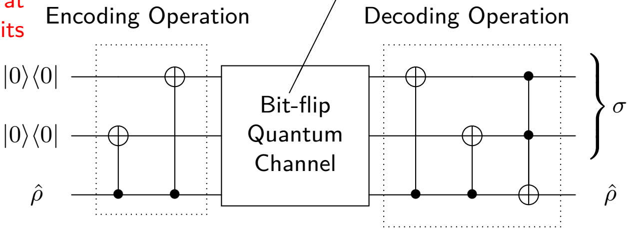

$\mathbf{V} = \text{span}\{|000\rangle, |111\rangle\}$

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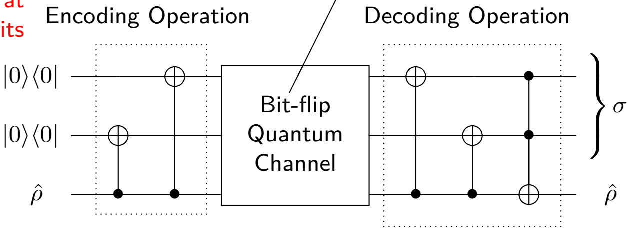

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[Nakahara, Tomita arXiv:1101.0413 (2011)]

Let $P_{\mathbf{V}} = |000\rangle\langle 000| + |111\rangle\langle 111|$. Then

$$P_{\mathbf{V}} F_i^\dagger F_j P_{\mathbf{V}} = \lambda_{ij} P_{\mathbf{V}} \quad \text{with} \quad [\lambda_{ij}] = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix}$$

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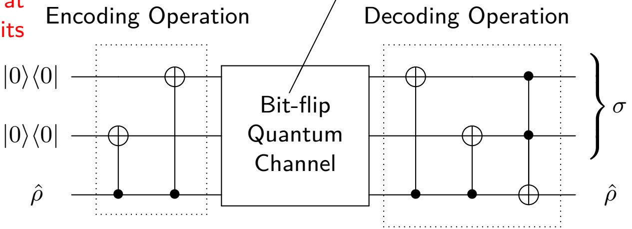

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$$\begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix} \in \Lambda_2 \begin{pmatrix} F_1 \dagger F_1 & F_1^\dagger F_2 & F_1^\dagger F_3 & F_1^\dagger F_4 \\ F_2 \dagger F_1 & F_2^\dagger F_2 & F_2^\dagger F_3 & F_2^\dagger F_4 \\ F_3 \dagger F_1 & F_3^\dagger F_2 & F_3^\dagger F_3 & F_3^\dagger F_4 \\ F_4 \dagger F_1 & F_4^\dagger F_2 & F_4^\dagger F_3 & F_4^\dagger F_4 \end{pmatrix}$$

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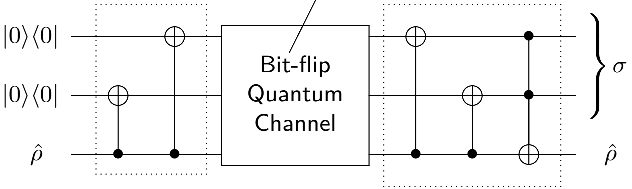
Encoding Operation

Decoding Operation



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$$\begin{aligned} |000\rangle &\longrightarrow |000\rangle \\ |001\rangle &\longrightarrow |111\rangle \end{aligned}$$

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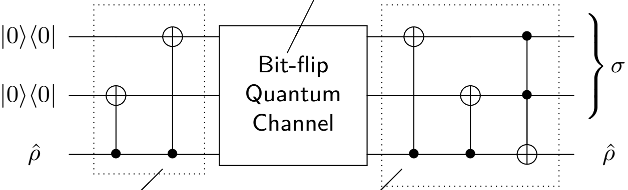

$$\mathbf{V} = \mathrm{span}\,\{|000\rangle, |111\rangle\}$$

[Nakahara, Tomita arXiv:1101.0413 (2011)]

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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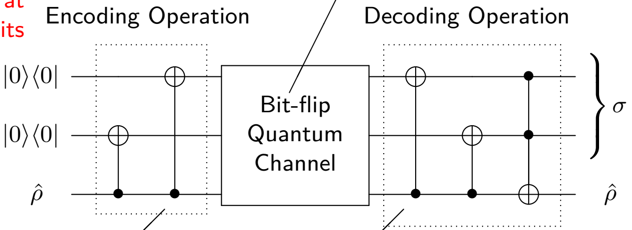

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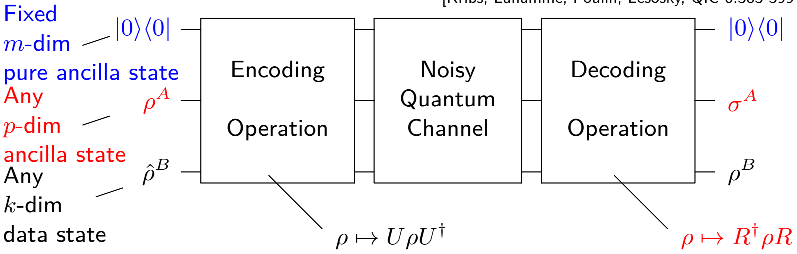
Operator Quantum Error Correction (OQEC)

[Kribs, Laflamme, Poulin, Lesosky, QIC 6:383-399 (2006)]



Decompose $\mathcal{H} = (A \otimes B) \oplus \mathcal{K}$

$$\forall \rho^A \in A, \rho^B \in B, \exists \sigma^A \in A, \quad \mathcal{E}\left(U\left(|0\rangle\langle 0| \otimes \rho^A \otimes \hat{\rho}\right)U^\dagger\right) = R\left(|0\rangle\langle 0| \otimes \sigma^A \otimes \hat{\rho}\right)R^\dagger$$

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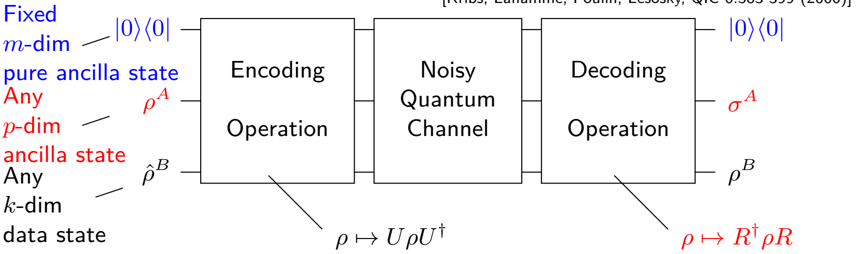

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Unitarily Recoverable Subsystem (URS) [Kribs, Spekkens, PRA 74:042329 (2006)]

A subsystem B of $A \otimes B$ is a recoverable subsystem for $\mathcal{E}$ if and only if

$$P_{rr}F_i^\dagger F_j P_{ss} = \lambda_{ijrs}P_{rs} \quad \forall 1 \le i,j \le p, 1 \le r,s \le p$$

where $P_{rs} = |r\rangle\langle s| \otimes I_B$ for $1 \le r,s \le p$.

(p, k) numerical range

Joint (p, k) numerical range

We define the joint (p, k) numerical (matricial) range of $\mathcal{A} = (A_1, \ldots, A_m)$ with $A_j \in \mathcal{B}(\mathcal{H})$ by

$$\Lambda_{p,k}(\mathcal{A}) = \{(B_1, \ldots, B_m) : \exists X \text{ s.t. } X^\dagger X = I_{pk} \text{ and } X^\dagger A_j X = B_j \otimes I_k\}$$

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The (p,k) numerical range of $A \in \mathcal{B}(\mathcal{H})$ is defined by

$$\begin{aligned} \Lambda_{p,k}(A) &= \{B \in M_p : \exists X \text{ s.t. } X^\dagger X = I_{pk} \text{ and } X^\dagger A X = B \otimes I_k\} \\ &= \left\{ B \in M_p : \exists \text{unitary } U \text{ s.t. } U^\dagger A U = \begin{bmatrix} B \otimes I_k & * \\ * & * \end{bmatrix} \right\}. \end{aligned}$$

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- When $p = k = 1$, we get the classical numerical range defined by

  $$W(A) = \{x^* A x : x \in \mathcal{H} \text{ with } \langle x, x \rangle = 1\}.$$

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An example

Consider the following bi-unitary channel $\mathcal{E}$ in M_8 defined by

$$\rho \mapsto t\rho + (1-t)U\rho U^\dagger \quad \text{where} \quad U = \operatorname{diag}(1, w, \ldots, w^7)$$

with $w = e^{\frac{2\pi i}{8}}$. Notice that $\mathcal{E}$ has a recoverable subsystem iff

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Let

$$X = \begin{bmatrix} a & 0 & 0 & 0 & \sqrt{1-a^2} & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 & \sqrt{1-b^2} \\ 0 & 0 & c & 0 & 0 & \sqrt{1-c^2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1-c^2} & 0 & 0 & c & 0 \end{bmatrix}^\dagger$$

with $a = \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}$, $b = \frac{1}{\sqrt{2}}$, and $c = \frac{1}{\sqrt{1+\sqrt{2}}}$.

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$$X^\dagger U X = \begin{bmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -b(1-c^2) & 0 \\ 0 & 0 & 0 & -b(1-c^2) \end{bmatrix} = \begin{bmatrix} b & 0 \\ 0 & -b(1-c^2) \end{bmatrix} \otimes I_2 = B \otimes I_2.$$

i.e., $B \in \Lambda_{2,2}(U)$.

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Three Qubit Fully Correlated Quantum Channel

A noisy quantum channel is called fully correlated when all the qubits constituting the codeword are subject to the same error operators.

$$\rho \mapsto \sum_{j=1}^{4} p_j F_j \rho F_j^\dagger \qquad F_j \in \left\{ I^{\otimes 3}, X^{\otimes 3}, Y^{\otimes 3}, Z^{\otimes 3} \right\}$$

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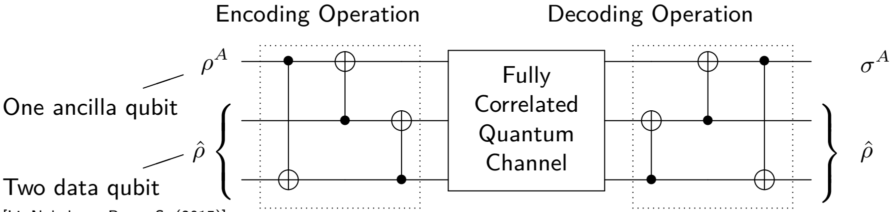

[Li, Nakahara, Poon, S. (2015)]

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$$\emptyset \neq \Lambda_{2,8} \begin{pmatrix} F_1^\dagger F_1 & F_1^\dagger F_2 & F_1^\dagger F_3 & F_1^\dagger F_4 \\ F_2^\dagger F_1 & F_2^\dagger F_2 & F_2^\dagger F_3 & F_2^\dagger F_4 \\ F_3^\dagger F_1 & F_3^\dagger F_2 & F_3^\dagger F_3 & F_3^\dagger F_4 \\ F_4^\dagger F_1 & F_4^\dagger F_2 & F_4^\dagger F_3 & F_4^\dagger F_4 \end{pmatrix} \quad \text{with} \quad F_j \in \left\{ I^{\otimes 5}, X^{\otimes 5}, Y^{\otimes 5}, Z^{\otimes 5} \right\}.$$

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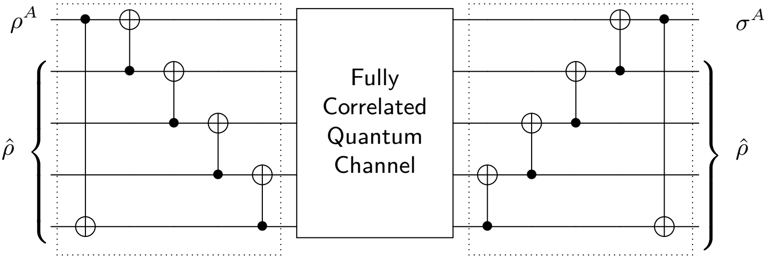

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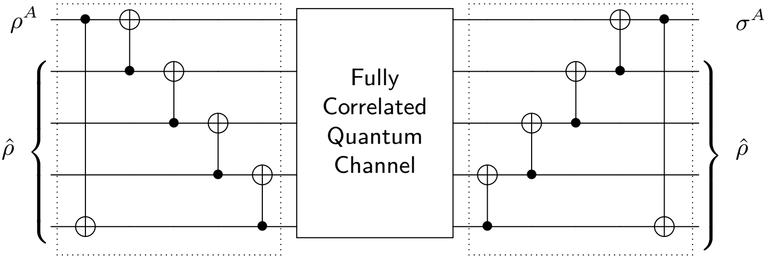

Fully Correlated Channel [Li, Nakahara, Poon, S. (2015)]

Odd n: one can encode $(n-1)$-data qubit states to n-qubit codewords with one ancilla qubit.

Three Qubit General Fully Correlated Quantum Channel

$$\rho \mapsto \sum_{j=1}^{r} p_j F_j \rho F_j^\dagger \qquad F_j \in \left\{ U^{\otimes n} : U \text{ is } 2 \times 2 \text{ unitary} \right\} = SU(2)^{\otimes n}$$

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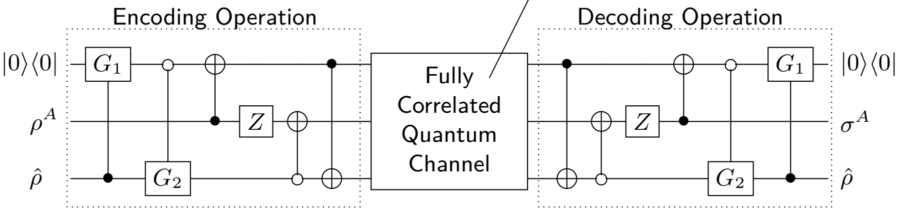

[Li, Nakahara, Poon, S., Tomita, PRA 84:044301 (2011)]

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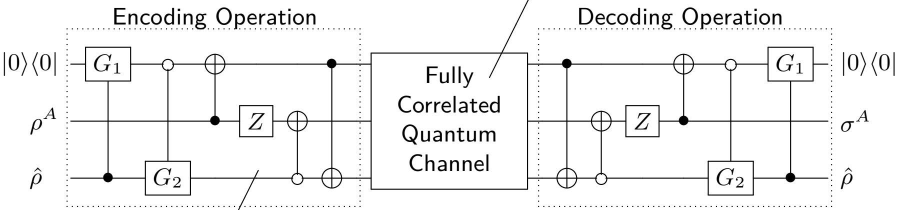

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{-2}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & 0 & 0 & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 0 & \frac{-1}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & 0 & 0 & 0 & \frac{-1}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$G_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}$$

$$G_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

[Li, Nakahara, Poon, S., Tomita, PRA 84:044301 (2011)]

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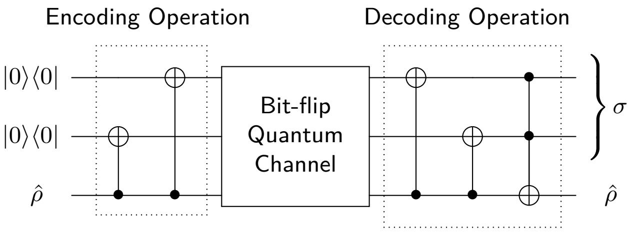
Three qubit bit-flip quantum channel

$$\mathcal{E} : \rho \mapsto F_1 \rho F_1^\dagger + F_2 \rho F_2^\dagger + F_3 \rho F_3^\dagger + F_4 \rho F_4^\dagger$$

$$\begin{aligned} F_1 &= \sqrt{q_1}\, I \otimes I \otimes I \\ F_2 &= \sqrt{q_2}\, X \otimes I \otimes I \\ F_3 &= \sqrt{q_3}\, I \otimes X \otimes I \\ F_4 &= \sqrt{q_4}\, I \otimes I \otimes X \end{aligned}$$



[Nakahara, Tomita arXiv:1101.0413 (2011)]

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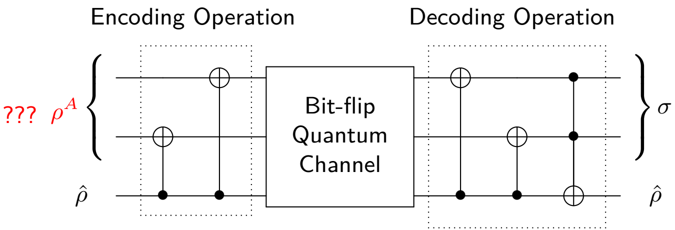

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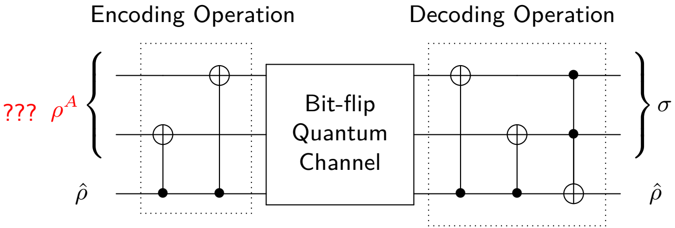

$$\Lambda_{4,2} \begin{pmatrix} F_1^\dagger F_1 & F_1^\dagger F_2 & F_1^\dagger F_3 & F_1^\dagger F_4 \\ F_2^\dagger F_1 & F_2^\dagger F_2 & F_2^\dagger F_3 & F_2^\dagger F_4 \\ F_3^\dagger F_1 & F_3^\dagger F_2 & F_3^\dagger F_3 & F_3^\dagger F_4 \\ F_4^\dagger F_1 & F_4^\dagger F_2 & F_4^\dagger F_3 & F_4^\dagger F_4 \end{pmatrix} = \emptyset$$

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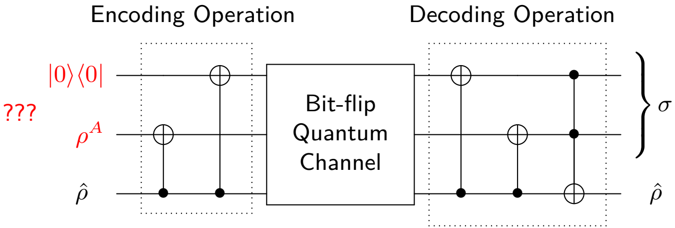

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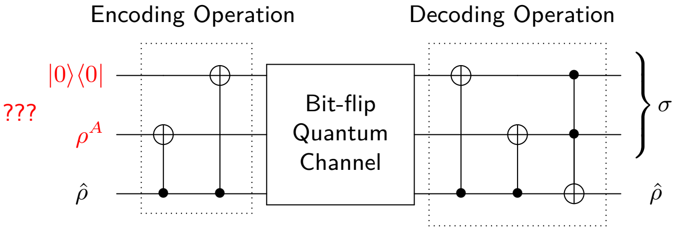

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Four qubit bit-flip quantum channel

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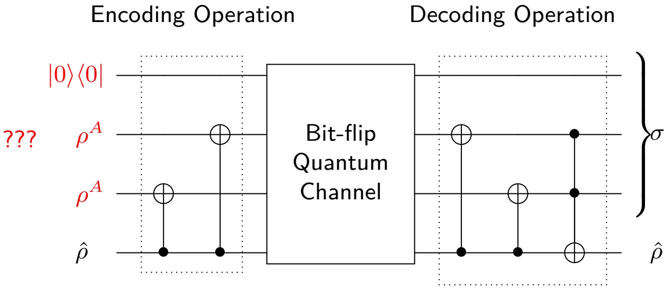

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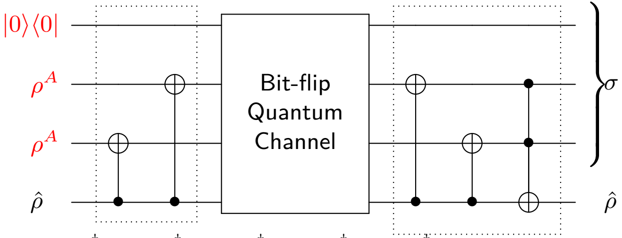
Encoding Operation   Decoding Operation

???



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Basic properties of $\Lambda_{p,k}(A)$:

- $\Lambda_{p,k}(\alpha A + \beta I_n) = \alpha\Lambda_{p,k}(A) + \beta I_p$ for any $\alpha, \beta \in \mathbb{C}$.
- $\Lambda_{p,k}(U^\dagger AU) = \Lambda_{p,k}(A)$ for any unitary U.
- $\Lambda_{p,k}(X^\dagger AX) \subseteq \Lambda_{p,k}(A)$ for any $n \times m$ matrix X with $X^\dagger X = I_m$.
- $B \in \Lambda_{p,k}(A) \quad \Longleftrightarrow \quad U^\dagger BU \in \Lambda_{p,k}(A)$ for all unitary U.

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- $B \in \Lambda_{p,k}(A) \iff U^\dagger B U \in \Lambda_{p,k}(A)$ for all unitary U.

A necessary condition

If $B \in \Lambda_{p,k}(A)$, then

$$\lambda_{n-(p-j+1)k+1}(\mathbf{Re}\,(e^{-it}A)) \le \lambda_j(\mathbf{Re}\,(e^{-it}B)) \le \lambda_{jk}(\mathbf{Re}\,(e^{-it}A))$$

for all $t \in [0, 2\pi)$ and $j = 1, \ldots, p$.

Here $\mathbf{Re}\,(X) = \dfrac{X + X^\dagger}{2}$ and $\lambda_j(X)$ is the jth largest eigenvalue of X.

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Hermitian case

Generalized interlacing inequalities

Suppose A is Hermitian. Then $B \in \Lambda_{p,k}(A)$ if and only if

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Non-emptyness and Convexity

Suppose A is Hermitian and $n \ge (p+1)k - 1$.

- $\Lambda_{p,k}(A)$ is always non-empty.
- $\Lambda_{p,k}(A)$ is convex if and only if $\lambda_{n-pk+1}(A) \le \lambda_{pk}(A)$.
- $\Lambda_{p,k}$ is a singleton, which is a scalar matrix, if and only if $\lambda_k(A) = \lambda_{n-k+1}(A)$.

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Empty or single unitary orbit

Suppose A is Hermitian and $n < (p+1)k - 1$.

- $\Lambda_{p,k}(A)$ is either empty or equal to a single unitary similarty orbit of a matrix $B \in M_p$ such that

$$\lambda_{n-(p-j+1)k+1}(A) = \lambda_j(B) = \lambda_{jk}(A) \text{ for all } j = 1, \ldots, p.$$

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Normal case

Non-emptyness

Let A be normal. If $n \geq (3k-2)p$, then $\Lambda_{p,k}(A)$ contains a normal matrix and hence is non-empty.

Proof. It suffices to show the case when $n = (3k-2)p$. As A is normal, we may assume $A = A_1 \oplus \cdots \oplus A_p$ where $A_j \in M_m$ with $m = 3k-2$. Then $\Lambda_k(A_j) \neq \emptyset$, and hence there is $m \times k$ matrix X_j with $X_j^\dagger X_j = I_k$ such that

$$X_j^\dagger A_j X_j = \lambda_j I_k \quad j = 1, \ldots, p.$$

Let $X = X_1 \oplus \cdots \oplus X_p$. Then

$$X^\dagger A X = \operatorname{diag}(\lambda_1, \ldots, \lambda_p) \otimes I_k.$$

i.e., $\operatorname{diag}(\lambda_1, \ldots, \lambda_p) \in \Lambda_{p,k}(A)$.

Normal case

An example which the range contains no normal matrix

Suppose $A = \text{diag}\,(1, w, \ldots, w^{n-1})$ where $w = e^{2\pi i/n}$ with $n = (3k-2)p-1$. Then $\Lambda_{p,k}(A)$ does not contain a normal matrix.

- For $n = 7$, $\Lambda_{2,2}(\text{diag}\,(1, \ldots, w^6))$ does not contain any normal matrix.

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Proposition

Suppose $n \leq (3k-2)p - 1$ and A is a normal matrix with eigenvalues $a_1, \ldots, a_n$ such that $\text{conv}\,\{a_1, \ldots, a_n\}$ is a n-sided polygon. Then $\Lambda_{p,k}(A)$ does not contain a normal matrix.

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- However, it is unknown whether $\Lambda_{p,k}(A)$ contains non-normal matrices or not.

An example which the range contains no normal matrix

Suppose $A = \mathrm{diag}\,(1, w, \ldots, w^{n-1})$ where $w = e^{2\pi i/n}$ with $n = (3k-2)p-1$. Then $\Lambda_{p,k}(A)$ does not contain a normal matrix.

- For $n = 7$, $\Lambda_{2,2}(\mathrm{diag}\,(1, \ldots, w^6))$ does not contain any normal matrix.

Proposition

Suppose $n \leq (3k-2)p-1$ and A is a normal matrix with eigenvalues $a_1, \ldots, a_n$ such that $\mathrm{conv}\,\{a_1, \ldots, a_n\}$ is a n-sided polygon. Then $\Lambda_{p,k}(A)$ does not contain a normal matrix.

- However, it is unknown whether $\Lambda_{p,k}(A)$ contains non-normal matrices or not.
- **Question**: Is it possible to find an example of a normal matrix A with $n = (3k-2)p-1$ such that $\Lambda_{p,k}(A) = \emptyset$?

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Non-emptyness

Suppose $A \in \mathcal{B}(\mathcal{H})$. If $n \geq 2(p+1)k - 3$, then $\Lambda_{p,k}(A)$ is non-empty.

General case

Non-emptyness

Suppose $A \in \mathcal{B}(\mathcal{H})$. If $n \geq 2(p+1)k-3$, then $\Lambda_{p,k}(A)$ is non-empty.

- In particular, $\Lambda_{2,2}(A)$ is always non-empty if $n \geq 9$.

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Non-emptyness

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- In particular, $\Lambda_{2,2}(A)$ is always non-empty if $n \geq 9$.
- It can be showed that $\Lambda_{2,2}(A)$ is always non-empty if $n = 8$.

General case

Non-emptyness

Suppose $A \in \mathcal{B}(\mathcal{H})$. If $n \geq 2(p+1)k-3$, then $\Lambda_{p,k}(A)$ is non-empty.

- In particular, $\Lambda_{2,2}(A)$ is always non-empty if $n \geq 9$.
- It can be showed that $\Lambda_{2,2}(A)$ is always non-empty if $n = 8$.
- **Question**: For $n = 7$, is it possible to find an example of A such that $\Lambda_{2,2}(A) = \emptyset$? Or is $\Lambda_{2,2}(A)$ always non-empty??
- **Question**: In general, what is the smallest dimension n such that $\Lambda_{p,k}(A)$ is always non-empty?

Joint (p, k) numerical range

Joint (p, k) numerical range

Given $\mathcal{A} = (A_1, \ldots, A_m)$ with Hermitian A_j. Define the joint (p, k) matricial range of $\mathcal{A}$ by

$$\Lambda_{p,k}(\mathcal{A}) = \{(B_1, \ldots, B_m) : \exists X \text{ s.t. } X^\dagger A_j X = B_j \otimes I_k \text{ and } X^\dagger X = I_k\}.$$

Non-emptyness

Suppose $\mathcal{A} = (A_1, \ldots, A_m)$ with Hermitian A_j. If $n \geq ((p+1)k - 2)m^2$, then $\Lambda_{p,k}(\mathcal{A}) \neq \emptyset$.

Question: What is the smallest dimension n such that $\Lambda_{p,k}(\mathcal{A})$ is always non-empty?

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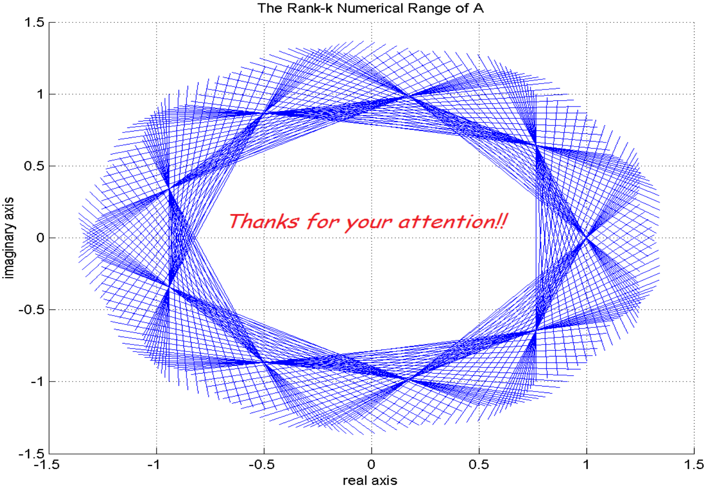

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